

# TWO PROPERTIES OF VOLUME GROWTH ENTROPY IN HILBERT GEOMETRY

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ABSTRACT. The aim of this paper is to provide two examples in Hilbert geometry which show that volume growth entropy may vanish for a *non*-polygonal domain in the plane on the one hand, and that volume growth entropy is *not* always a limit on the other hand.

## 1. INTRODUCTION

A *Hilbert domain* in  $\mathbf{R}^m$  is a metric space  $(\mathcal{C}, d_{\mathcal{C}})$ , where  $\mathcal{C}$  is an *open bounded convex* set in  $\mathbf{R}^m$  and  $d_{\mathcal{C}}$  is the distance function on  $\mathcal{C}$  — called the *Hilbert metric* — defined as follows.

Given two distinct points  $p$  and  $q$  in  $\mathcal{C}$ , let  $a$  and  $b$  be the intersection points of the straight line defined by  $p$  and  $q$  with  $\partial\mathcal{C}$  so that  $p = (1 - s)a + sb$  and  $q = (1 - t)a + tb$  with  $0 < s < t < 1$ . Then

$$d_{\mathcal{C}}(p, q) \doteq \frac{1}{2} \ln[a, p, q, b],$$

where

$$[a, p, q, b] \doteq \frac{1 - s}{s} \times \frac{t}{1 - t} > 1$$

is the cross ratio of the 4-tuple of ordered collinear points  $(a, p, q, b)$ .

We complete the definition by setting  $d_{\mathcal{C}}(p, p) \doteq 0$ .

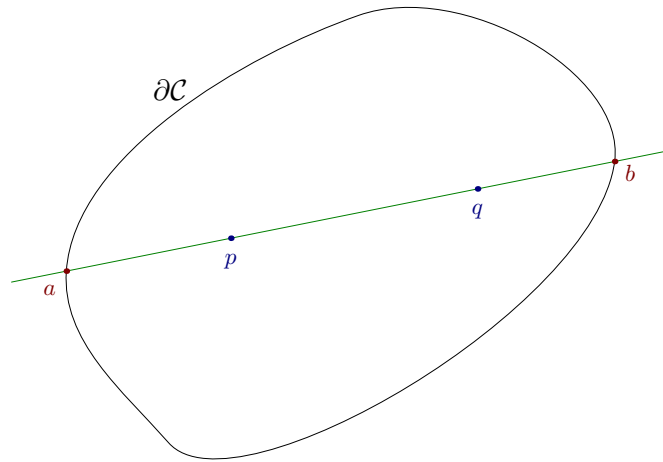


FIGURE 1. The Hilbert metric  $d_{\mathcal{C}}$

The metric space  $(\mathcal{C}, d_{\mathcal{C}})$  thus obtained is a complete non-compact geodesic metric space whose topology is the one induced by the canonical topology of  $\mathbf{R}^m$  and in which the affine open segments joining two points of the boundary  $\partial\mathcal{C}$  are geodesic lines. It is to be mentioned here that in general the affine segment between two points in  $\mathcal{C}$  may *not* be the *unique* geodesic

joining these points (for example, if  $\mathcal{C}$  is a square). Nevertheless, this uniqueness holds whenever  $\mathcal{C}$  is *strictly* convex.

Moreover, the distance function  $d_{\mathcal{C}}$  is associated with the Finsler metric  $F_{\mathcal{C}}$  on  $\mathcal{C}$  given, for any  $p \in \mathcal{C}$  and any  $v \in T_p\mathcal{C} \equiv \mathbf{R}^m$  (the tangent vector space to  $\mathcal{C}$  at  $p$ ), by

$$F_{\mathcal{C}}(p, v) \doteq \frac{1}{2} \left( \frac{1}{t^-} + \frac{1}{t^+} \right) \quad \text{if } v \neq 0,$$

where  $t^- = t_{\mathcal{C}}^-(p, v)$  and  $t^+ = t_{\mathcal{C}}^+(p, v)$  are the *unique positive* numbers such that  $p - t^-v \in \partial\mathcal{C}$  and  $p + t^+v \in \partial\mathcal{C}$ , and  $F_{\mathcal{C}}(p, 0) \doteq 0$ .

**Remark.** For  $p \in \mathcal{C}$  and  $v \in T_p\mathcal{C} \equiv \mathbf{R}^m$  with  $v \neq 0$ , we will define  $p^- = p_{\mathcal{C}}^-(p, v) \doteq p - t_{\mathcal{C}}^-(p, v)v$  and  $p^+ = p_{\mathcal{C}}^+(p, v) \doteq p + t_{\mathcal{C}}^+(p, v)v$ . Then, given any arbitrary norm  $\|\cdot\|$  on  $\mathbf{R}^m$ , we can write

$$F_{\mathcal{C}}(p, v) = \frac{1}{2} \|v\| \left( \frac{1}{\|p - p^-\|} + \frac{1}{\|p - p^+\|} \right).$$

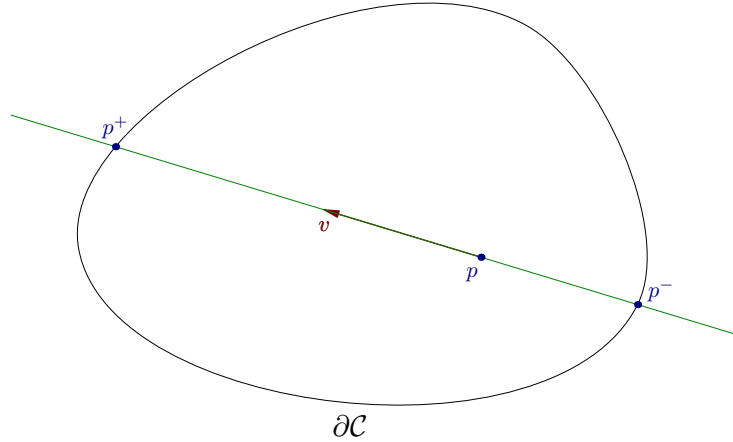


FIGURE 2. The Finsler metric  $F_{\mathcal{C}}$

Finally, let  $\text{vol}$  be the canonical Lebesgue measure on  $\mathbf{R}^m$  and define  $\omega_m \doteq \text{vol}(\mathbf{B}^m)$ .

For  $p \in \mathcal{C}$ , let  $B_{\mathcal{C}}(p) \doteq \{v \in \mathbf{R}^m \mid F_{\mathcal{C}}(p, v) < 1\}$  be the unit open ball with respect to the norm  $F_{\mathcal{C}}(p, \cdot)$  on  $T_p\mathcal{C} \equiv \mathbf{R}^m$ .

The measure  $\mu_{\mathcal{C}}$  on  $\mathcal{C}$  associated with the Finsler metric  $F_{\mathcal{C}}$  is then defined, for any Borel set  $A \subseteq \mathcal{C}$ , by

$$\mu_{\mathcal{C}}(A) \doteq \int_A \frac{\omega_m}{\text{vol}(B_{\mathcal{C}}(p))} d\text{vol}(p)$$

and will be called the *Hilbert measure* associated with  $(\mathcal{C}, d_{\mathcal{C}})$ .

**Remark.** The Borel measure  $\mu_{\mathcal{C}}$  is the classical Busemann measure of the Finsler space  $(\mathcal{C}, F_{\mathcal{C}})$  and corresponds to the Hausdorff measure of the metric space  $(\mathcal{C}, d_{\mathcal{C}})$  (see [3, page 199, Example 5.5.13]).

Thanks to this measure, we can make use of a concept of fundamental importance, the *volume growth entropy*, which is attached to any metric space. Very often, this notion is introduced for

Let  $(\mathcal{C}, d_{\mathcal{C}})$  be a Hilbert domain in  $\mathbf{R}^m$  admitting a cocompact group of isometries, for which we may assume  $0 \in \mathcal{C}$  since translations in  $\mathbf{R}^m$  preserve the cross ratio.

If for any  $R > 0$  we denote by  $\mathcal{B}_{\mathcal{C}}(0, R) \doteq \{p \in \mathcal{C} \mid d_{\mathcal{C}}(0, p) < R\}$  the open ball of radius  $R$  about 0 in  $(\mathcal{C}, d_{\mathcal{C}})$ , then the volume growth entropy of  $d_{\mathcal{C}}$  writes

$$h(\mathcal{C}) \doteq \lim_{R \rightarrow +\infty} \frac{1}{R} \ln[\mu_{\mathcal{C}}(\mathcal{B}_{\mathcal{C}}(0, R))].$$

Now, when we drop cocompactness, it is not known whether this limit still exists, as stated in [10, question raised in section 2.5].

Therefore, the main goal of this paper is to answer to this question and to show that the answer is *negative*.

**Main Theorem.** *There exists a Hilbert domain  $(\mathcal{C}, d_{\mathcal{C}})$  in  $\mathbf{R}^2$  that satisfies*

$$\limsup_{R \rightarrow +\infty} \frac{1}{R} \ln[\mu_{\mathcal{C}}(\mathcal{B}_{\mathcal{C}}(0, R))] > 0,$$

and

$$\liminf_{R \rightarrow +\infty} \frac{1}{R} \ln[\mu_{\mathcal{C}}(\mathcal{B}_{\mathcal{C}}(0, R))] = 0.$$

**Remark.** However, it is to be noticed that these two limits coincide not only for Hilbert domains having a cocompact group of isometries as mentioned above, but also in the case when the boundary  $\partial\mathcal{C}$  of  $\mathcal{C}$  is strongly convex (see [9]) together with the case when  $\mathcal{C}$  is a polytope (see [15]).

The proof of this theorem will be given in the last section by constructing an explicit example which is a convex ‘polygon’ with *infinitely* many vertices having an accumulation point around which the boundary of the ‘polygon’ strongly looks like a circle.

The intuitive idea behind this construction is that, depending on where we are located in the ‘polygon’, its boundary may look like the one of a usual polygon — and hence the volume growth entropy behaves as if it were vanishing — (this corresponds to the  $\liminf$  part in the theorem) or like a small portion of a circle (around the accumulation point) — and hence the volume growth entropy behaves as if it were positive — (this corresponds to the  $\limsup$  part in the theorem).

On the other hand, using such ‘polygons’ with infinitely many vertices, we show that there exist *non*-polygonal Hilbert domains whose volume growth entropy is zero. This is stated in Theorem 3.2.

For further information about Hilbert geometry, we refer to [4, 5, 11, 12, 14] and the excellent introduction [13] by Socié-Méthou.

About the importance of volume growth and topological entropies in Hilbert geometry, we may have a look at the interesting work [10] by Crampon and the references therein.

## 2. PRELIMINARIES

This section is devoted to recalling a result by Berck, Bernig and Vernicos ([1]) that is useful for our purpose, and on the other hand to proving a technical result we will need throughout the present work.

**Definition 2.1.** A subset  $Y$  of  $\mathbf{R}^m$  will be called a *pseudo-hypersurface* of  $\mathbf{R}^m$  if and only if

- (1)  $Y$  is a topological hypersurface of  $\mathbf{R}^m$ , and
- (2) there exists a Borel set  $Z$  in  $Y$  such that
  - (a)  $Z$  has measure zero, and
  - (b)  $Y \setminus Z$  is a  $C^1$  hypersurface of  $\mathbf{R}^m$ .

**Remark 2.1.** Recall that having measure zero for a Borel set in a *topological manifold* is an intrinsic notion that does *not* depend on any Borel measure. On the other hand, being of class  $C^1$  at a point of a topological manifold is a local property, and hence  $Y \setminus Z$  is open in  $Y$ , that is,  $Z$  is closed in  $Y$ .

Given a pseudo-hypersurface  $Y$  of  $\mathbf{R}^m$  such that  $Y \subseteq \mathcal{C}$ , let us denote by  $\lambda_{\mathcal{C}}(Y)$  the  $n$ -dimensional Hausdorff measure of  $Y$  associated with  $d_{\mathcal{C}}$ .

This number does *not* of course depend on the Borel set  $Z$  arising in Definition 2.1 and is equal to the Busemann measure of  $Y \setminus Z$  associated with the restriction of the Finsler metric  $F_{\mathcal{C}}$  to  $Y \setminus Z$ .

Now, if for any  $R > 0$  we denote by  $\mathcal{S}_{\mathcal{C}}(0, R) := \{p \in \mathcal{C} \mid d_{\mathcal{C}}(0, p) = R\}$  the sphere of radius  $R$  about 0 in the metric space  $(\mathcal{C}, d_{\mathcal{C}})$ , then we have the following result proved in [1]:

**Proposition 2.1.** *The following properties hold:*

- (1) *the boundary  $\partial \mathcal{C}$  is a pseudo-hypersurface of  $\mathbf{R}^m$ ;*
- (2) *for all  $t > 0$ , the sphere  $\mathcal{S}_{\mathcal{C}}(0, t) \subseteq \mathcal{C}$  is a pseudo-hypersurface of  $\mathbf{R}^m$ ;*
- (3) *the volume growth entropy of  $d_{\mathcal{C}}$  is equal to*

$$h(\mathcal{C}) = \limsup_{R \rightarrow +\infty} \frac{1}{R} \ln[\lambda_{\mathcal{C}}(\mathcal{S}_{\mathcal{C}}(0, R))].$$

**Notations.** From now on, the canonical Euclidean distance between any two points  $p$  and  $q$  in  $\mathbf{R}^2$  will be denoted by  $pq$ , the canonical Lebesgue measure  $\text{vol}$  on  $\mathbf{R}^2$  by  $\mathcal{A}$  (area), and the open convex hull of any three points  $a$ ,  $b$  and  $c$  in  $\mathbf{R}^2$  by  $abc$  (open triangle).

In addition, the spherical distance between two vectors  $u \neq 0$  and  $v \neq 0$  in  $\mathbf{R}^2$  will be denoted by  $\angle(u, v)$  (defined as the unique  $\theta \in [0, \pi]$  such that  $\cos \theta = \langle u/\|u\|, v/\|v\| \rangle \in [-1, 1]$ , where  $\langle \cdot, \cdot \rangle$  and  $\|\cdot\|$  stand for the canonical Euclidean scalar product and the canonical Euclidean norm on  $\mathbf{R}^2$ , respectively).

With these notations, here is the second key ingredient we will use in the sequel:

**Proposition 2.2.** *Let  $(\mathcal{C}, d_{\mathcal{C}})$  be a Hilbert domain in  $\mathbf{R}^2$  with  $0 \in \mathcal{C}$  and  $A, P, Q, B$  be points*

- (i)  $(\overrightarrow{0\check{A}}, \overrightarrow{0\check{B}})$ ,  $(\overrightarrow{0\check{A}}, \overrightarrow{0\check{P}})$ ,  $(\overrightarrow{0\check{P}}, \overrightarrow{0\check{Q}})$  and  $(\overrightarrow{0\check{Q}}, \overrightarrow{0\check{B}})$  are bases of the vector space  $\mathbf{R}^2$  with the same orientation,
- (ii)  $A$  and  $B$  do not belong to the straight line  $(PQ)$ , and
- (iii) the affine segments  $[A, P]$ ,  $[P, Q]$  and  $[P, B]$  are in the boundary  $\partial\mathcal{C}$ .

Denote by  $\check{P}$  and  $\check{Q}$  the intersection points of the boundary  $\partial\mathcal{C}$  with the half-lines  $\mathbf{R}_+\overrightarrow{0\check{P}}$  and  $\mathbf{R}_+\overrightarrow{0\check{Q}}$ , respectively.

Then for any  $R > 0$  the points  $p, q \in \mathcal{C}$  defined by  $p \in [0, P]$ ,  $q \in [0, Q]$  and  $d_{\mathcal{C}}(0, p) = d_{\mathcal{C}}(0, q) = R$  satisfy the following properties:

- (1) the half-lines  $p + \mathbf{R}_+\overrightarrow{pq}$  and  $P + \mathbf{R}_+\overrightarrow{P\check{A}}$  meet if and only if one has  $e^{2R} > \tau_A$ , where

$$\tau_A := \frac{\mathcal{A}(P0Q)}{\mathcal{A}(\check{P}0\check{Q})} \times \frac{\mathcal{A}(A0\check{Q}) - \mathcal{A}(A0\check{P}) - \mathcal{A}(P0\check{Q})}{\mathcal{A}(PAQ)};$$

- (2) the half-lines  $q + \mathbf{R}_+\overrightarrow{pq}$  and  $Q + \mathbf{R}_+\overrightarrow{Q\check{B}}$  meet if and only if one has  $e^{2R} > \tau_B$ , where

$$\tau_B := \frac{\mathcal{A}(P0Q)}{\mathcal{A}(\check{P}0\check{Q})} \times \frac{\mathcal{A}(B0\check{P}) - \mathcal{A}(B0\check{Q}) - \mathcal{A}(Q0\check{P})}{\mathcal{A}(PBQ)};$$

- (3) the half-line  $p + \mathbf{R}_+\overrightarrow{pq}$  intersects with the segment  $[A, P]$  if and only if  $R \geq \rho_A$  holds, where

$$\rho_A := \frac{1}{2} \ln \left( \frac{\mathcal{A}(P0Q)}{\mathcal{A}(\check{P}0\check{Q})} \times \frac{\mathcal{A}(\check{P}A\check{Q})}{\mathcal{A}(PAQ)} \right);$$

- (4) the half-line  $q + \mathbf{R}_+\overrightarrow{pq}$  intersects with the segment  $[B, Q]$  if and only if  $R \geq \rho_B$  holds, where

$$\rho_B := \frac{1}{2} \ln \left( \frac{\mathcal{A}(P0Q)}{\mathcal{A}(\check{P}0\check{Q})} \times \frac{\mathcal{A}(\check{P}B\check{Q})}{\mathcal{A}(PBQ)} \right);$$

- (5) whenever  $R \geq \rho := \max\{\rho_A, \rho_B\}$ , we have

$$e^{2d_{\mathcal{C}}(p,q)} = \left( 1 + \frac{\mathcal{A}(Apq) + \mathcal{A}(Ppq)}{\mathcal{A}(ApP)} \right) \times \left( 1 + \frac{\mathcal{A}(Bqp) + \mathcal{A}(Qqp)}{\mathcal{A}(BqQ)} \right).$$

### Remarks.

- 1) In Points (1) and (2), the constants  $\tau_A$  and  $\tau_B$  may be non-positive.
- 2) In Points (3) and (4), if we have  $A \in (\check{P}\check{Q})$  or  $B \in (\check{P}\check{Q})$ , then we get  $\rho_A = -\infty$  or  $\rho_B = -\infty$ , respectively.
- 3) As we shall see in the proof of Points (3) and (4), one always has  $\tau_A < e^{2\rho_A}$  and  $\tau_B < e^{2\rho_B}$ .

### Proof of Proposition 2.2.

Let  $\alpha = \angle(\overrightarrow{0\check{A}}, \overrightarrow{0\check{P}})$ ,  $\theta = \angle(\overrightarrow{0\check{P}}, \overrightarrow{0\check{Q}})$  and  $\beta = \angle(\overrightarrow{0\check{Q}}, \overrightarrow{0\check{B}})$ , and define  $\alpha_P = \angle(\overrightarrow{P\check{0}}, \overrightarrow{P\check{A}})$ ,  $\theta_p = \angle(\overrightarrow{p\check{0}}, \overrightarrow{p\check{q}})$ ,  $\theta_q = \angle(\overrightarrow{q\check{0}}, \overrightarrow{q\check{p}})$  and  $\beta_Q = \angle(\overrightarrow{Q\check{0}}, \overrightarrow{Q\check{B}})$  (see Figure 3).

It is to be noticed that we have  $0 < \theta$ ,  $0 < \alpha \leq \pi - \theta$  and  $0 < \beta \leq \pi - \theta$  by Assumption (i) (which yields  $\alpha, \beta, \theta \in (0, \pi)$ ). Moreover, Assumption (i) implies that the canonical Euclidean distances  $AP$ ,  $pq$  and  $BQ$  do not vanish.

Now, applying the sine law in the triangles  $0AP$  and  $0pq$ , we have

$$0A \quad \quad \quad 0q$$

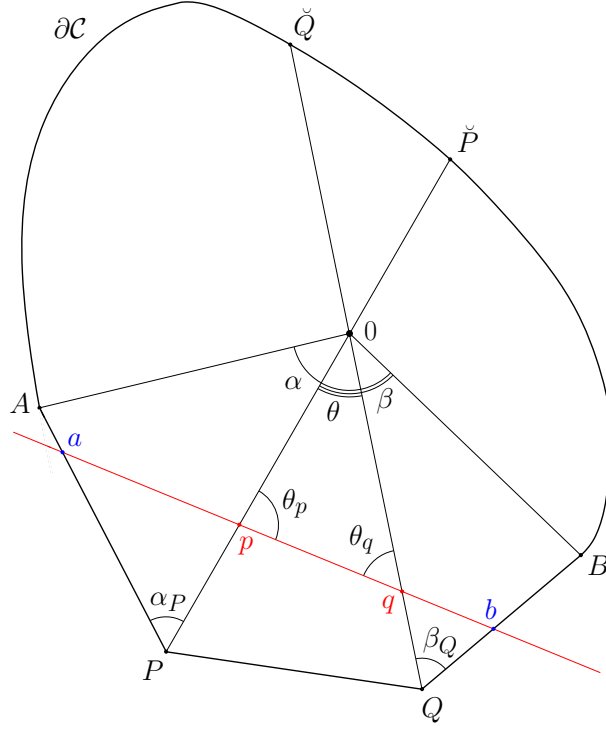


FIGURE 3. Proposition 2.2

(Passing by, this shows that  $\alpha_P$  and  $\theta_p$  are in  $(0, \pi)$ .)

On the other hand, considering the triangle  $0AP$ , the cosine law yields

$2 \times 0A \times 0P \times \cos \alpha = 0A^2 + 0P^2 - AP^2$  and  $2 \times AP \times 0P \times \cos \alpha_P = AP^2 + 0P^2 - 0A^2$ ,  
from which we get (by adding these two equations)

$$(2.2) \quad \cos \alpha_P = \frac{0P - 0A \times \cos \alpha}{AP}.$$

In a similar way, considering the triangle  $0pq$ , one has

$$(2.3) \quad \cos \theta_p = \frac{0p - 0q \times \cos \theta}{pq}.$$

Therefore, combining Equation 2.1, Equation 2.2 and Equation 2.3 with the formula

$$\sin(\alpha_P + \theta_p) = \sin \alpha_P \times \cos \theta_p + \sin \theta_p \times \cos \alpha_P,$$

we can write

$$(2.4) \quad \sin(\alpha_P + \theta_p) = \frac{0A \times 0p \times \sin \alpha + 0P \times 0q \times \sin \theta - 0A \times 0q \times \sin(\alpha + \theta)}{AP \times pq}.$$

Applying the same arguments as above in triangles  $0BQ$  and  $0qp$  also gives

$$(2.5) \quad \sin(\beta_Q + \theta_q) = \frac{0B \times 0q \times \sin \beta + 0Q \times 0p \times \sin \theta - 0B \times 0p \times \sin(\beta + \theta)}{BQ \times qp}.$$

•Point (1): The half-lines  $p + \mathbf{R}_+ \vec{qp}$  and  $P + \mathbf{R}_+ \vec{PA}$  meet if and only if

$$\angle(\vec{PA}, \vec{Pp}) + \angle(\vec{pP}, \vec{qp}) < \pi,$$

In other words, the half-lines  $p + \mathbf{R}_+ \overrightarrow{qp}$  and  $P + \mathbf{R}_+ \overrightarrow{PA}$  meet if and only if  $\sin(\alpha_P + \theta_p)$  is positive since one already has  $\alpha_P + \theta_p > 0$  (see the remark made after Equation 2.1).

Using Equation 2.4, this writes

$$(2.6) \quad 0A \times 0p \times \sin \alpha + 0P \times 0q \times \sin \theta > 0A \times 0q \times \sin(\alpha + \theta).$$

But

$$e^{2R} = e^{2dc(0,p)} = [\check{P}, 0, p, P] = \frac{\check{P}p}{\check{P}0} \times \frac{P0}{Pp} = \frac{\check{P}0 + 0p}{\check{P}0} \times \frac{0P}{0P - 0p} = \frac{0P}{0\check{P}} \times \frac{0\check{P} + 0p}{0P - 0p},$$

or equivalently

$$(2.7) \quad 0p = \frac{0P \times 0\check{P} \times (e^{2R} - 1)}{0P + 0\check{P} \times e^{2R}},$$

and the same holds for  $q$ , *i. e.*,

$$(2.8) \quad 0q = \frac{0Q \times 0\check{Q} \times (e^{2R} - 1)}{0Q + 0\check{Q} \times e^{2R}}.$$

Hence, from Equation 2.7 and Equation 2.8, Condition 2.6 is the same as

$$\begin{aligned} & 0P \times 0Q \times (0A \times 0\check{Q} \times \sin(\alpha + \theta) - 0A \times 0\check{P} \times \sin \alpha - 0P \times 0\check{Q} \times \sin \theta) \\ & < 0\check{P} \times 0\check{Q} \times (0A \times 0P \times \sin \alpha + 0P \times 0Q \times \sin \theta - 0A \times 0Q \times \sin(\alpha + \theta)) e^{2R}, \end{aligned}$$

or

$$(2.9) \quad \mathcal{A}(A0\check{Q}) - \mathcal{A}(A0\check{P}) - \mathcal{A}(P0\check{Q}) < \frac{\mathcal{A}(\check{P}0\check{Q})}{\mathcal{A}(P0\check{Q})} \left( \mathcal{A}(A0P) + \mathcal{A}(P0Q) - \mathcal{A}(A0Q) \right) e^{2R}$$

since we can write  $0A \times 0P \times \sin \alpha = 2\mathcal{A}(A0P)$ ,  $0P \times 0Q \times \sin \theta = 2\mathcal{A}(P0Q)$ , etc. (according to the sine law), and  $0A \times 0\check{P} \times \sin \alpha = 2\mathcal{A}(A0\check{P})$ ,  $0P \times 0\check{Q} \times \sin \theta = 2\mathcal{A}(P0\check{Q})$ , etc. (using  $\angle(\overrightarrow{0A}, \overrightarrow{0P}) = \pi - \angle(\overrightarrow{0A}, \overrightarrow{0\check{P}}) = \pi - \alpha$ ,  $\angle(\overrightarrow{0P}, \overrightarrow{0Q}) = \pi - \angle(\overrightarrow{0P}, \overrightarrow{0\check{Q}}) = \pi - \theta$ , etc.) together with  $0P \times 0Q = 2\mathcal{A}(P0Q) / \sin \theta$  and  $0\check{P} \times 0\check{Q} = 2\mathcal{A}(\check{P}0\check{Q}) / \sin \theta$  (indeed,  $\angle(\overrightarrow{0P}, \overrightarrow{0Q}) = \angle(\overrightarrow{0P}, \overrightarrow{0\check{Q}}) = \theta$ ). Now, the convexity of  $\mathcal{C}$  implies that the points  $0$ ,  $A$ ,  $P$  and  $Q$  are in the boundary of their convex hull in  $\mathbf{R}^2$ , and hence  $\mathcal{A}(A0P) + \mathcal{A}(P0Q) - \mathcal{A}(A0Q) = \mathcal{A}(PAQ)$ . Using this equality in Equation 2.9 proves Point (1) in Proposition 2.2.

•**Point (2):** It is obtained exactly the same way as previously by replacing  $A$  by  $B$ ,  $\alpha$  by  $\beta$ ,  $\theta_p$  by  $\theta_q$  and  $\alpha_P$  by  $\beta_Q$ , and by using Equation 2.5.

•**Point (3):** Suppose  $e^{2R} > \tau_A$ , and let  $a$  be the intersection point of  $p + \mathbf{R}_+ \overrightarrow{qp}$  with  $P + \mathbf{R}_+ \overrightarrow{PA}$ . Notice that  $Pa > 0$ ; indeed, if this were not the case, then  $q \in (ap) = (Pp) = (0P)$ , which is false since  $(\overrightarrow{0P}, \overrightarrow{0Q})$  is a basis of  $\mathbf{R}^2$ .

Since  $\angle(\overrightarrow{Pp}, \overrightarrow{Pa}) = \angle(\overrightarrow{P0}, \overrightarrow{PA}) = \alpha_P$  and  $\angle(\overrightarrow{pP}, \overrightarrow{pa}) = \angle(\overrightarrow{p0}, \overrightarrow{pq}) = \theta_p$ , the sine law in the triangle  $Pap$  then yields

$$ap = Pp \times \frac{\sin \alpha_P}{\sin(\pi - (\alpha_P + \theta_p))} = (0P - 0p) \times \frac{\sin \alpha_P}{\sin(\alpha_P + \theta_p)},$$

that is,

$$(2.10) \quad ap = \frac{(0P - 0p) \times 0A \times pq \times \sin \alpha}{0A \times 0p \times \sin \alpha + 0P \times 0q \times \sin \theta - 0A \times 0q \times \sin(\alpha + \theta)}$$

In addition, still in the triangle  $Pap$ , the sine law together with Equation 2.1 give

$$Pa = ap \times \frac{\sin \theta_p}{\sin \alpha_P} = \frac{ap \times 0q \times AP \times \sin \theta}{0A \times pq \times \sin \alpha},$$

and hence

$$(2.11) \quad Pa = \frac{(0P - 0p) \times 0q \times AP \times \sin \theta}{0A \times 0p \times \sin \alpha + 0P \times 0q \times \sin \theta - 0A \times 0q \times \sin(\alpha + \theta)}$$

by Equation 2.10.

Now, the point  $a$  belongs to the segment  $[A, P]$  if and only if  $Pa \leq PA$ , *i. e.*, using Equation 2.11,

$$0A \times 0q \times \sin(\alpha + \theta) \leq 0A \times 0p \times \sin \alpha + 0p \times 0q \times \sin \theta,$$

which is equivalent to

$$(2.12) \quad 0P \times 0Q \times \left( 0A \times 0\check{Q} \times \sin(\alpha + \theta) + 0\check{P} \times 0\check{Q} \times \sin \theta - 0A \times 0\check{P} \times \sin \alpha \right) \\ \leq 0\check{P} \times 0\check{Q} \times \left( 0A \times 0P \times \sin \alpha + 0P \times 0Q \times \sin \theta - 0A \times 0Q \times \sin(\alpha + \theta) \right) e^{2R},$$

by Equation 2.7 and Equation 2.8.

Since the convexity of  $\mathcal{C}$  implies that the points  $0, A, P$  and  $Q$  (respectively  $0, A, \check{Q}$  and  $\check{P}$ ) are in the boundary of their convex hull in  $\mathbf{R}^2$ , we have  $\mathcal{A}(A0P) + \mathcal{A}(P0Q) - \mathcal{A}(A0Q) = \mathcal{A}(PAQ)$  (respectively  $\mathcal{A}(A0\check{Q}) + \mathcal{A}(\check{P}0\check{Q}) - \mathcal{A}(A0\check{P}) = \mathcal{A}(\check{P}A\check{Q})$ ), and hence writing  $0A \times 0P \times \sin \alpha = 2\mathcal{A}(A0P)$ ,  $0P \times 0Q \times \sin \theta = 2\mathcal{A}(P0Q)$ , etc. in Equation 2.12 yields

$$(2.13) \quad e^{2R} \geq \frac{\mathcal{A}(P0Q)}{\mathcal{A}(\check{P}0\check{Q})} \times \frac{\mathcal{A}(\check{P}A\check{Q})}{\mathcal{A}(PAQ)} = e^{2\rho_A}.$$

Finally, using again the fact that the points  $0, A, \check{Q}$  and  $\check{P}$  are in the boundary of their convex hull in  $\mathbf{R}^2$ , we have  $\mathcal{A}(A0\check{Q}) \leq \mathcal{A}(A0\check{P}) + \mathcal{A}(\check{P}A\check{Q})$ . Hence

$$\mathcal{A}(A0\check{Q}) - \mathcal{A}(A0\check{P}) - \mathcal{A}(P0\check{Q}) < \mathcal{A}(A0\check{Q}) - \mathcal{A}(A0\check{P}) \leq \mathcal{A}(\check{P}A\check{Q}),$$

and this implies that  $\tau_A < e^{2\rho_A}$  from the very definitions of  $\tau_A$  and  $\rho_A$  given in Proposition 2.2.

Combining this latter inequality with Equation 2.13 shows Point (3) in Proposition 2.2.

•**Point (4):** On the other hand, if we suppose  $e^{2R} > \tau_B$  and let  $b$  be the intersection point of  $q + \mathbf{R}_+ \overrightarrow{pq}$  with  $Q + \mathbf{R}_+ \overrightarrow{QB}$ , then the same reasoning as above yields Point (4) in Proposition 2.2.

•**Point (5):** Suppose  $R \geq \rho = \max\{\rho_A, \rho_B\}$ .

Writing

$$e^{2dc(p,q)} = [a, p, q, b] = \frac{aq}{ap} \times \frac{bp}{bq} = \left( \frac{ap + pq}{ap} \right) \times \left( \frac{bq + qp}{bq} \right) = \left( 1 + \frac{pq}{ap} \right) \times \left( 1 + \frac{pq}{bq} \right)$$

and using Equation 2.10 together with its analogue

$$bq = \frac{(0Q - 0q) \times 0B \times pq \times \sin \beta}{0B \times 0q \times \sin \beta + 0Q \times 0p \times \sin \theta - 0B \times 0p \times \sin(\beta + \theta)},$$

we obtain

$$e^{2dc(p,q)} = \left( 1 + \frac{0A \times 0p \times \sin \alpha + 0P \times 0q \times \sin \theta - 0A \times 0q \times \sin(\alpha + \theta)}{0A \times 0P \times \sin \alpha - 0A \times 0p \times \sin \alpha} \right) \\ \times \left( 1 + \frac{0B \times 0q \times \sin \beta + 0Q \times 0p \times \sin \theta - 0B \times 0p \times \sin(\beta + \theta)}{0B \times 0Q \times \sin \beta - 0B \times 0p \times \sin \beta} \right)$$



In other words,

$$(2.14) \quad e^{2d_C(p,q)} = \left(1 + \frac{\mathcal{A}(A0p) + \mathcal{A}(P0q) - \mathcal{A}(A0Q)}{\mathcal{A}(A0P) - \mathcal{A}(A0p)}\right) \times \left(1 + \frac{\mathcal{A}(B0q) + \mathcal{A}(Q0p) - \mathcal{A}(B0P)}{\mathcal{A}(B0Q) - \mathcal{A}(B0q)}\right).$$

Now, since  $a \in [A, P]$  (respectively  $b \in [B, Q]$ ), the points  $0, A, p$  and  $q$  (respectively  $0, B, q$  and  $p$ ) are in the boundary of their convex hull in  $\mathbf{R}^2$ , and therefore  $\mathcal{A}(A0p) + \mathcal{A}(P0q) - \mathcal{A}(A0Q) = \mathcal{A}(Apq) + \mathcal{A}(Ppq)$  (respectively  $\mathcal{A}(B0q) + \mathcal{A}(Q0p) - \mathcal{A}(B0P) = \mathcal{A}(Bqp) + \mathcal{A}(Qqp)$ ).

On the other hand,  $p \in [0, P]$  (respectively  $q \in [0, Q]$ ) obviously implies that  $\mathcal{A}(A0P) - \mathcal{A}(A0p) = \mathcal{A}(ApP)$  (respectively  $\mathcal{A}(B0P) - \mathcal{A}(B0q) = \mathcal{A}(BqQ)$ ).

Hence Equation 2.14 is equivalent to the equality in Point (5) of Proposition 2.2.  $\square$

### 3. NON-POLYGONAL DOMAINS MAY HAVE ZERO ENTROPY

In this section, we construct a Hilbert domain in the plane which is a ‘polygon’ having infinitely many vertices and whose volume growth entropy is a limit that is equal to zero. This ‘polygon’ is inscribed in a circle and its vertices have one accumulation point.

Before giving our example, let us first recall the following result proved in [15]:

**Theorem 3.1.** *Given any open convex polytope  $\mathcal{P}$  in  $\mathbf{R}^m$  that contains the origin  $0$ , the volume growth entropy of  $d_{\mathcal{P}}$  satisfies*

$$h(\mathcal{P}) = \lim_{R \rightarrow +\infty} \frac{1}{R} \ln[\mu_{\mathcal{P}}(\mathcal{B}_{\mathcal{P}}(0, R))] = 0.$$

**Remark.** Another — but less direct — proof of this theorem consists in saying that  $(\mathcal{P}, d_{\mathcal{P}})$  is Lipschitz equivalent to Euclidean plane as shown in [2] (and in [8] for the particular case when  $n = 2$ ), and hence  $h(\mathcal{P}) = 0$  since the volume growth entropy of any finite-dimensional normed vector space is equal to zero.

Now, let us show that having zero volume growth entropy for a Hilbert domain in  $\mathbf{R}^2$  does *not* mean being polygonal, that is, that the converse of Theorem 3.1 is *false*.

Let  $(P_n)_{n \in \mathbf{N}}$  be the sequence of points in  $\mathbf{S}^1$  defined by

$$P_n = (\cos(2^{-n}), \sin(2^{-n})),$$

and denote by  $\mathcal{C}$  the open convex hull in  $\mathbf{R}^2$  of the set

$$\{P_n, -P_n \mid n \in \mathbf{N}\}.$$

Then we have

**Theorem 3.2.** *The volume growth entropy of  $d_{\mathcal{C}}$  satisfies*

$$h(\mathcal{C}) = \lim_{R \rightarrow +\infty} \frac{1}{R} \ln[\mu_{\mathcal{C}}(\mathcal{B}_{\mathcal{C}}(0, R))] = 0.$$

**Remark.** More precisely, we will show in the proof of this result that the volume  $\mu_{\mathcal{C}}(\mathcal{B}_{\mathcal{C}}(0, R))$

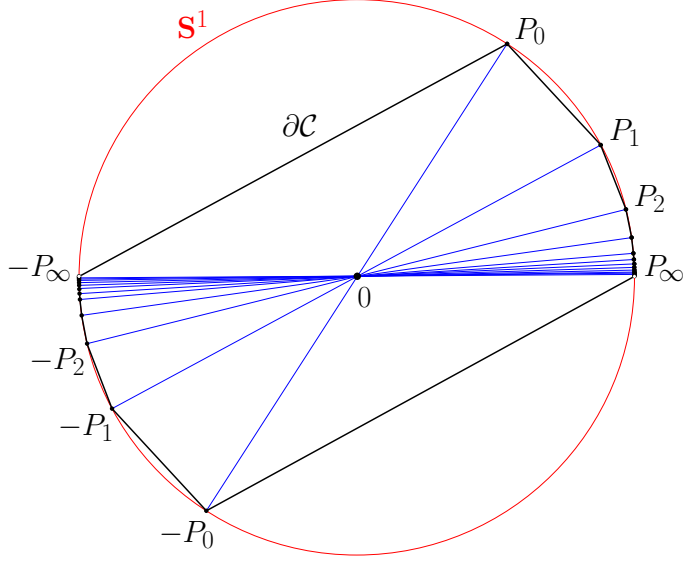


FIGURE 4. A non-polygonal Hilbert domain in the plane with zero entropy

In order to prove this result, we need the following:

**Lemma 3.1.** *Let  $(\mathcal{C}, d_{\mathcal{C}})$  be a Hilbert domain in  $\mathbf{R}^2$  with  $0 \in \mathcal{C}$ , and  $P, Q$  be distinct points in  $\partial\mathcal{C}$  such that the affine segment  $[P, Q]$  is in the boundary  $\partial\mathcal{C}$ .*

*Denote respectively by  $\check{P}$  and  $\check{Q}$  the intersection points of the boundary  $\partial\mathcal{C}$  with the half-lines  $\mathbf{R}_-\vec{0\check{P}}$  and  $\mathbf{R}_-\vec{0\check{Q}}$ .*

*If  $[\check{P}, \check{Q}] \subseteq \partial\mathcal{C}$ , then for any  $R > 0$  the intersections of  $\mathcal{B}_{\mathcal{C}}(0, R)$  and  $\mathcal{S}_{\mathcal{C}}(0, R)$  with the triangle  $P0Q$  are equal to the triangle  $p0q$  and the open segment  $]p, q[$ , respectively, where the points  $p$  and  $q$  are defined in  $\mathcal{C}$  by  $p \in [0, P]$ ,  $q \in [0, Q]$  and  $d_{\mathcal{C}}(0, p) = d_{\mathcal{C}}(0, q) = R$ .*

*Proof.*

Denote by  $\Delta$  and  $\check{\Delta}$  the straight lines  $(PQ)$  and  $(\check{P}\check{Q})$ , respectively.

Since the cross ratio is invariant under  $\text{PGL}(\mathbf{R}^3)$ , we may assume that  $\Delta$  and  $\check{\Delta}$  are not parallel (replacing  $\mathcal{C}$  by its image by a projective transformation), and hence consider their intersection point  $\Omega$ .

Then the four lines  $\Delta$ ,  $\check{\Delta}$ ,  $(\Omega 0)$  and  $(\Omega p)$  define a pencil with common vertex  $\Omega$ , which implies that for any point  $m \in P0Q$  we have the equivalence

$$(3.1) \quad [M, m, 0, \check{M}] = [P, p, 0, \check{P}] \iff m \in (\Omega p),$$

where  $M$  and  $\check{M}$  are the intersection points of  $(0m)$  with  $\Delta$  and  $\check{\Delta}$ , respectively.

Since one has  $[P, p, 0, \check{P}] = e^{2d_{\mathcal{C}}(0, p)} = e^{2R}$ , Equivalence 3.1 then yields

$$\mathcal{S}_{\mathcal{C}}(0, R) \cap P0Q = \{m \in P0Q \mid [M, m, 0, \check{M}] = e^{2R}\} = P0Q \cap (\Omega p).$$

But  $[Q, q, 0, \check{Q}] = e^{2d_{\mathcal{C}}(0, q)} = e^{2R}$  implies  $q \in (\Omega p)$  by Equivalence 3.1, and hence  $P0Q \cap (\Omega p) = ]p, q[$  since we have  $q \in [0, Q]$ .

Next, writing  $\mathcal{B}_{\mathcal{C}}(0, R) = \bigcup_{r \in [0, R)} \mathcal{S}_{\mathcal{C}}(0, r)$ , we get  $\mathcal{B}_{\mathcal{C}}(0, R) \cap P0Q = p0q$ .

This proves Lemma 3.1. □

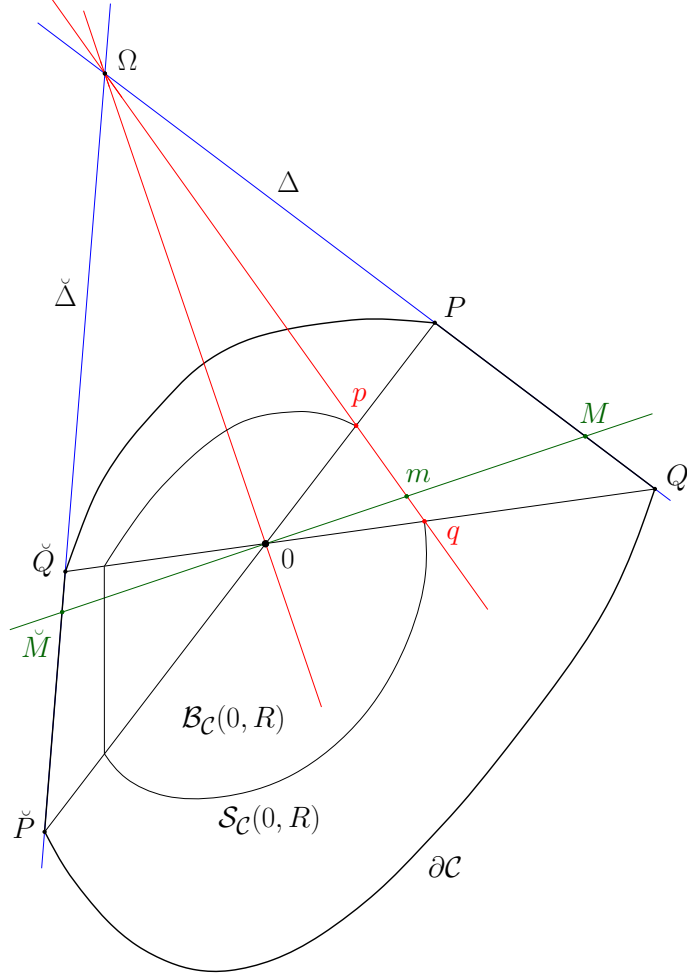


FIGURE 5. Lemma 3.1

*Proof of Theorem 3.2.*

For each  $k \in \mathbf{N}$ , let  $\mathcal{T}_k$  be the open rectangle that is equal to the open convex hull in  $\mathbf{R}^2$  of  $P_k$ ,  $-P_k$ ,  $P_{k+1}$  and  $-P_{k+1}$ .

Fixing an integer  $n \geq 0$  and a number  $R \geq 1$ , we can write

$$\begin{aligned}
 (3.2) \quad \frac{1}{2} \mu_C(\mathcal{B}_C(0, R)) &= \mu_C(\mathcal{B}_C(0, R) \cap -P_\infty 0 P_0) \\
 &+ \sum_{k=0}^n \mu_C(\mathcal{B}_C(0, R) \cap P_k 0 P_{k+1}) \\
 &+ \mu_C(\mathcal{B}_C(0, R) \cap S(P_{n+1} 0 P_\infty))
 \end{aligned}$$

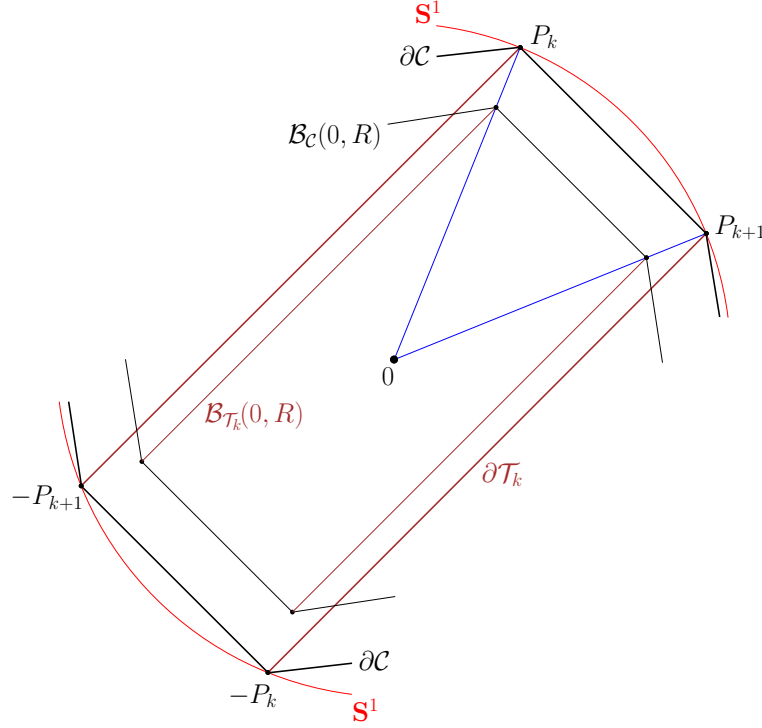
with  $P_\infty \asymp (1, 0) = \lim_{k \rightarrow +\infty} P_k \in \partial \mathcal{C}$  and where  $S(P_{n+1} 0 P_\infty)$  denotes the sector defined as the convex hull of the union of the half-lines  $\mathbf{R}_+ 0 \overrightarrow{P_{n+1}}$  and  $\mathbf{R}_+ 0 \overrightarrow{P_\infty}$ .

•**First step:** For any  $k \in \mathbf{N}$ , noticing that  $[P_k, P_{k+1}] \subseteq \partial \mathcal{C} \cap \partial \mathcal{T}_k$  and  $[-P_k, -P_{k+1}] \subseteq \partial \mathcal{C} \cap \partial \mathcal{T}_k$ , we have

$$\mathcal{B}_C(0, R) \cap P_k 0 P_{k+1} = \mathcal{B}_{\mathcal{T}_k}(0, R) \cap P_k 0 P_{k+1}$$

by Lemma 3.1, and hence the inclusions

$$\mathcal{B}_C(0, R) \cap P_k 0 P_{k+1} \subseteq \mathcal{B}_{\mathcal{T}_k}(0, R) \subseteq \mathcal{T}_k \subseteq \mathcal{C}$$

FIGURE 6. Comparing  $\mathcal{B}_C(0, R) \cap P_k 0 P_{k+1}$  and  $\mathcal{B}_{T_k}(0, R) \cap P_k 0 P_{k+1}$ 

Using Point (iv) of Proposition 5 in [7], we then get

$$(3.3) \quad \mu_C(\mathcal{B}_C(0, R) \cap P_k 0 P_{k+1}) \leq \mu_{T_k}(\mathcal{B}_{T_k}(0, R)).$$

Now, if  $f$  denotes the unique linear transformation of  $\mathbf{R}^2$  such that  $f(P_k) = (1, -1)$  and  $f(P_{k+1}) = (1, 1)$ , we have

$$f(\mathcal{T}_k) = \mathcal{Q} \doteq (-1, 1) \times (-1, 1) \subseteq \mathbf{R}^2 \quad (\text{standard open square}).$$

The cross ratio being preserved by the linear group  $\text{GL}(\mathbf{R}^2)$ , the map  $f$  induces an isometry between the metric spaces  $(\mathcal{T}_k, d_{T_k})$  and  $(\mathcal{Q}, d_{\mathcal{Q}})$  with  $f(0) = 0$ , and hence we obtain  $\mu_{T_k}(\mathcal{B}_{T_k}(0, R)) = \mu_{\mathcal{Q}}(\mathcal{B}_{\mathcal{Q}}(0, R))$ .

But Proposition 6 in [7] yields

$$\mu_{\mathcal{Q}}(\mathcal{B}_{\mathcal{Q}}(0, R)) = 8 \int_0^{\tanh(R)} \left( \int_0^x \frac{\pi}{\text{vol}(B_C(x, y))} dy \right) dx \leq 4 \int_0^{\tanh(R)} \left( \int_0^x \frac{\pi}{(1-x^2)(1-y^2)} dy \right) dx = 2\pi R^2,$$

which gives

$$(3.4) \quad \mu_C(\mathcal{B}_C(0, R) \cap P_k 0 P_{k+1}) \leq 2\pi R^2$$

from Equation 3.3.

For the same reasons, we also have

$$(3.5) \quad \mu_C(\mathcal{B}_C(0, R) \cap -P_{\infty} 0 P_0) \leq 2\pi R^2.$$

•**Second step:** Considering the proof of Theorem 12 in [6], we have

$$\frac{1}{\text{vol}(B_C(p))} \leq \frac{e^{8R}}{\text{vol}(\mathcal{B}_C(0, R))}$$

Therefore, the following is true:

$$\begin{aligned}
\mu_C(\mathcal{B}_C(0, R) \cap S(P_{n+1}0P_\infty)) &\leq \frac{\pi e^{8R}}{\text{vol}(\mathcal{B}_C(0, R))} \text{vol}(\mathcal{B}_C(0, R) \cap S(P_{n+1}0P_\infty)) \\
&\leq \frac{\pi e^{8R}}{\text{vol}(\mathcal{B}_C(0, 1))} \text{vol}(\mathbf{B}^2 \cap S(P_{n+1}0P_\infty)) \\
&\quad (\text{using } \mathcal{B}_C(0, 1) \subseteq \mathcal{B}_C(0, R) \subseteq \mathbf{B}^2) \\
(3.6) \quad &= \frac{\pi e^{8R}}{\text{vol}(\mathcal{B}_C(0, 1))} \sum_{k=n+1}^{+\infty} \sin(2^{-k} - 2^{-(k+1)}) \\
&= \frac{\pi e^{8R}}{\text{vol}(\mathcal{B}_C(0, 1))} \sum_{k=n+1}^{+\infty} \sin(2^{-(k+1)}) \\
&\leq \frac{\pi e^{8R}}{\text{vol}(\mathcal{B}_C(0, 1))} \sum_{k=n+1}^{+\infty} 2^{-(k+1)} = \tau e^{8R}/2^n,
\end{aligned}$$

where  $\tau \doteq \pi/(2\text{vol}(\mathcal{B}_C(0, 1)))$  is a positive constant.

So, if we choose  $n \doteq [12R] + 1$  (where  $[\cdot]$  denotes the integer part), we have  $e^{8R}/2^n \leq 1$ , and hence Equation 3.6 implies

$$(3.7) \quad \mu_C(\mathcal{B}_C(0, R) \cap S(P_{n+1}0P_\infty)) \leq \tau.$$

•**Third step:** Combining Equations 3.2, 3.4, 3.5 and 3.7, we eventually get

$$\begin{aligned}
\mu_C(\mathcal{B}_C(0, R)) &\leq 4\pi R^2 + 4\pi(n+1)R^2 + \tau \\
&\leq 4\pi R^2 + 4\pi(12R+2)R^2 + \tau \\
&\quad (\text{since } n-1 = [12R] \leq 12R) \\
&\leq (144\pi + \tau)R^3
\end{aligned}$$

for any  $R \geq 1$ , and hence

$$\frac{\ln[\mu_C(\mathcal{B}_C(0, 1))]}{R} \leq \frac{\ln[\mu_C(\mathcal{B}_C(0, R))]}{R} \leq \frac{\ln((144\pi + \tau)R^3)}{R},$$

which yields  $\frac{\ln[\mu_C(\mathcal{B}_C(0, R))]}{R} \rightarrow 0$  as  $R \rightarrow +\infty$ .

This proves Theorem 3.2. □

#### 4. ENTROPY MAY NOT BE A LIMIT

We now come to the main goal of this paper which is to show that the volume growth entropy for a Hilbert domain may *not* be a limit. To this end, we will approximate a disc in the plane by an inscribed ‘polygonal’ domain with infinitely many vertices that have one accumulation point around which the boundary of the ‘polygonal’ domain looks very strongly like the boundary of the disc.

Let  $(n_k)_{k \geq 0}$  be the sequence of positive integers defined by

$$n_0 \doteq 3 \quad \text{and} \quad \forall k \geq 0, \quad n_{k+1} = 3^{n_k^2}.$$

Next, define the sequences  $(\alpha_k)_{k \geq 0}$  and  $(\theta_k)_{k \geq 0}$  in  $\mathbf{R}$  by  $\alpha_k \doteq 2\pi/n_k$  together with

$$\theta_0 \doteq 0 \quad \text{and} \quad \forall k \geq 1, \quad \theta_k \doteq \sum_{\ell=0}^{k-1} \alpha_\ell = 2\pi \sum_{\ell=0}^{k-1} \frac{1}{n_\ell}.$$

Finally, consider the sequence  $(M_k)_{k \geq 0}$  and the family  $(P_k(j))_{(k,j) \in \{(k,j) \in \mathbf{Z}^2 \mid k \geq 0 \text{ and } 0 \leq j \leq n_k\}}$  of points in  $\mathbf{S}^1$  defined by

$$M_k \doteq (\cos(\theta_k), \sin(\theta_k)) \quad \text{and} \quad P_k(j) \doteq (\cos(\theta_k + \alpha_k j/n_k), \sin(\theta_k + \alpha_k j/n_k)),$$

and denote by  $\mathcal{C}$  the open convex hull in  $\mathbf{R}^2$  of the set

$$\{P_k(j), -P_k(j) \mid k \geq 0 \text{ and } 0 \leq j \leq n_k\}.$$

Then we get the following:

**Theorem 4.1.** *We have*

$$(1) \quad h(\mathcal{C}) \doteq \limsup_{R \rightarrow +\infty} \frac{1}{R} \ln[\mu_{\mathcal{C}}(\mathcal{B}_{\mathcal{C}}(0, R))] > 0, \text{ and}$$

$$(2) \quad \liminf_{R \rightarrow +\infty} \frac{1}{R} \ln[\mu_{\mathcal{C}}(\mathcal{B}_{\mathcal{C}}(0, R))] = 0.$$

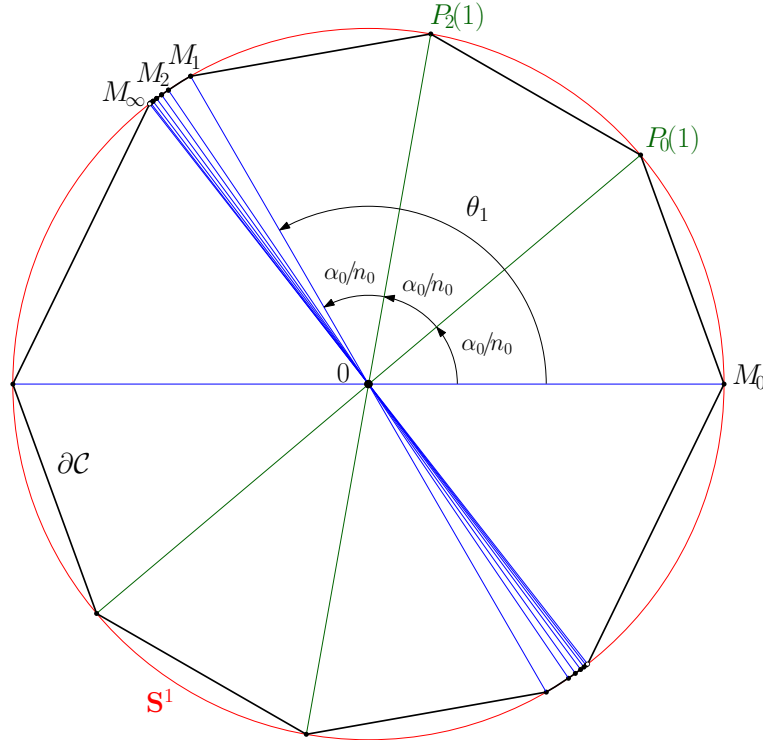


FIGURE 7. A Hilbert domain in the plane whose entropy is not a limit

**Remarks.**

2) For all  $\ell \geq 0$ , we have  $n_\ell \geq 3^{\ell+1}$  (by induction and using  $9^m \geq m$  for all integer  $m \geq 0$ ), and hence the increasing sequence  $(\theta_k)_{k \geq 0}$  converges to some real number  $\theta_\infty$  which satisfies

$$0 < \theta_\infty < \pi \text{ (since } \sum_{\ell=0}^{+\infty} 1/3^{\ell+1} = 1/3 \sum_{\ell=0}^{+\infty} (1/3)^\ell = 1/2 \text{ and } (n_\ell)_{\ell \geq 0} \neq (3^{\ell+1})_{\ell \geq 0} \text{)}.$$

*Proof of Theorem 4.1.*

•**Point (1):** Consider the sequence of positive numbers  $(r_k)_{k \geq 0}$  defined by  $r_k := \ln(n_k^2)$ .

Fix  $k \geq 0$ , and let  $(p_k(j))_{1 \leq j \leq n_k-1}$  be the sequence of points in  $\mathbf{R}^2$  defined by

$$p_k(j) \in [0, P_k(j)] \quad \text{and} \quad d_{\mathcal{C}}(0, p_k(j)) = r_k.$$

Then fix  $j \in \{1, \dots, n_k-2\}$ , and let  $A := P_k(j-1)$ ,  $P := P_k(j)$ ,  $Q := P_k(j+1)$  and  $B := P_k(j+2)$ .

These latter points and  $\mathcal{C}$  satisfy all the three assumptions of Proposition 2.2, and we have  $\check{P} = -P$  and  $\check{Q} = -Q$ .

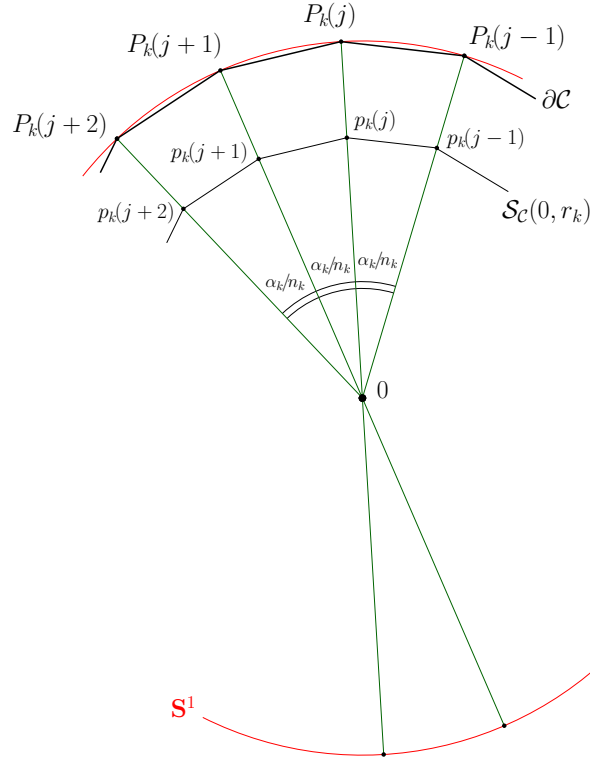


FIGURE 8. Showing  $\limsup_{k \rightarrow +\infty} \frac{\ln[\mu_{\mathcal{C}}(\mathcal{B}_{\mathcal{C}}(0, r_k))]}{r_k} > 0$  with  $r_k := \ln(n_k^2)$

Using Points (3) and (4) in Proposition 2.2 together with the equalities

$$\begin{aligned} 2\mathcal{A}(PBQ) &= 2\mathcal{A}(PAQ) = 2\mathcal{A}(A0P) + 2\mathcal{A}(P0Q) - 2\mathcal{A}(A0Q) \\ &= \sin(\alpha_k/n_k) + \sin(\alpha_k/n_k) - \sin(2\alpha_k/n_k) \\ &= 2\sin(\alpha_k/n_k)(1 - \cos(\alpha_k/n_k)) \end{aligned}$$

and

$$\begin{aligned}
2\mathcal{A}(\check{P}B\check{Q}) = 2\mathcal{A}(\check{P}A\check{Q}) &= 2\mathcal{A}(A0\check{Q}) + 2\mathcal{A}(\check{P}0\check{Q}) - 2\mathcal{A}(A0\check{P}) \\
&= \sin(\pi - 2\alpha_k/n_k) + \sin(\alpha_k/n_k) - \sin(\pi - \alpha_k/n_k) \\
&= \sin(2\alpha_k/n_k) \\
&= 2\sin(\alpha_k/n_k)\cos(\alpha_k/n_k) \\
&= 2\sin(2\pi/n_k^2)\cos(2\pi/n_k^2),
\end{aligned}$$

we first get

$$(4.1) \quad e^{2\rho_A} = e^{2\rho_B} = e^{2\rho} = \frac{\cos(2\pi/n_k^2)}{1 - \cos(2\pi/n_k^2)} \sim n_k^4/(2\pi^2) \quad \text{as } k \longrightarrow +\infty.$$

Then we deduce  $e^{2\rho}/n_k^4 \longrightarrow 1/(2\pi^2) < 1$  as  $k \longrightarrow +\infty$  from Equation 4.1, which means that there exists an integer  $k_0 \geq 0$  such that for all  $k \geq k_0$  one has  $e^{2\rho}/n_k^4 \leq 1$ , i. e.,  $r_k \geq \rho$ .

So, whenever  $k \geq k_0$ , Point (5) in Proposition 2.2 with  $p \doteq p_k(j)$  and  $q \doteq p_k(j+1)$  yields

$$(4.2) \quad e^{2d_C(p_k(j), p_k(j+1))} = e^{4r_k} \left(1 - (1 + e^{-2r_k}) \cos(2\pi/n_k^2)\right)^2 = n_k^8 \left(1 - (1 + 1/n_k^4) \cos(2\pi/n_k^2)\right)^2$$

since we have

$$\begin{aligned}
2(\mathcal{A}(Bqp) + \mathcal{A}(Qqp)) = 2(\mathcal{A}(Apq) + \mathcal{A}(Ppq)) &= 2\mathcal{A}(A0p) + 2\mathcal{A}(P0q) - 2\mathcal{A}(A0Q) \\
&= 0p \times \sin(\alpha_k/n_k) + 0q \times \sin(\alpha_k/n_k) - \sin(2\alpha_k/n_k) \\
&= 2\sin(\alpha_k/n_k)(\tanh(r_k) - \cos(\alpha_k/n_k)) \\
&= 2\sin(2\pi/n_k^2)(\tanh(r_k) - \cos(2\pi/n_k^2))
\end{aligned}$$

and

$$\begin{aligned}
2\mathcal{A}(BqQ) = 2\mathcal{A}(ApP) &= 2\mathcal{A}(A0P) - 2\mathcal{A}(A0p) \\
&= \sin(\alpha_k/n_k) - 0p \times \sin(\alpha_k/n_k) \\
&= (1 - \tanh(r_k)) \sin(\alpha_k/n_k) \\
&= (1 - \tanh(r_k)) \sin(2\pi/n_k^2)
\end{aligned}$$

by Equation 2.7 and Equation 2.8 (which give  $0p = 0q = \tanh(r_k)$ ).

But we have  $n_k^8(1 - (1 + 1/n_k^4) \cos(2\pi/n_k^2))^2 \longrightarrow (2\pi^2 - 1)^2 > e$  as  $k \longrightarrow +\infty$ , and hence there exists an integer  $k_1 \geq k_0$  such that for all  $k \geq k_1$  one has  $e^{2d_C(p_k(j), p_k(j+1))} \geq e$  by Equation 4.2, i. e.,

$$d_C(p_k(j), p_k(j+1)) \geq 1/2.$$

By Lemma 3.1, we then get

$$\lambda_C(\mathcal{S}_C(0, r_k)) \geq \sum_{j=1}^{n_k-2} d_C(p_k(j), p_k(j+1)) \geq n_k/2 - 1,$$

and hence

$$\frac{\ln[\lambda_C(\mathcal{S}_C(0, r_k))]}{r_k} \geq \frac{\ln(n_k/2 - 1)}{2\ln(n_k)}$$

for all  $k \geq k_1$ .

Finally, since  $\ln(n_k/2 - 1)/\ln(n_k) \longrightarrow 1 > 1/2$  as  $k \longrightarrow +\infty$ , there exists an integer  $k_2 \geq k_1$  such that for all  $k \geq k_2$  we have

$$\frac{\ln[\lambda_C(\mathcal{S}_C(0, r_k))]}{r_k} \geq \frac{\ln(n_k/2 - 1)}{2\ln(n_k)}$$



Since  $\partial\mathcal{C}$  is a pseudo-hypersurface of  $\mathbf{R}^2$ , Point (3) in Proposition 2.1 gives the first point of Theorem 4.1.

•Point (2): Consider the sequence of positive numbers  $(R_i)_{i \geq 0}$  defined by  $R_i := n_i$ .

On the other hand, for each  $k \geq 0$  and  $j \in \{0, \dots, n_k - 1\}$ , let  $\mathcal{T}_k(j)$  be the open rectangle that is equal to the open convex hull in  $\mathbf{R}^2$  of  $P_k(j)$ ,  $-P_k(j)$ ,  $P_k(j+1)$  and  $-P_k(j+1)$ .

Fixing an integer  $i \geq 0$ , we can write

$$\begin{aligned}
\frac{1}{2}\mu_{\mathcal{C}}(\mathcal{B}_{\mathcal{C}}(0, R_i)) &= \mu_{\mathcal{C}}(\mathcal{B}_{\mathcal{C}}(0, R_i) \cap -M_{\infty}0M_0) \\
&+ \sum_{k=0}^i \sum_{j=0}^{n_k-1} \mu_{\mathcal{C}}(\mathcal{B}_{\mathcal{C}}(0, R_i) \cap P_k(j)0P_k(j+1)) \\
&+ \mu_{\mathcal{C}}(\mathcal{B}_{\mathcal{C}}(0, R_i) \cap S(M_{i+1}0M_{\infty}))
\end{aligned}
\tag{4.3}$$

with  $M_\infty \coloneqq (\cos(\theta_\infty), \sin(\theta_\infty)) \in \partial\mathcal{C}$  (recall that  $\theta_\infty$  is the limit of the sequence  $(\theta_k)_{k \in \mathbf{N}}$ : see the second remark following Theorem 4.1) and where  $\overrightarrow{S(M_{i+1}0M_\infty)}$  denotes the sector defined as the convex hull of the union of the half-lines  $\mathbf{R}_+0M_{i+1}$  and  $\mathbf{R}_+0M_\infty$ .

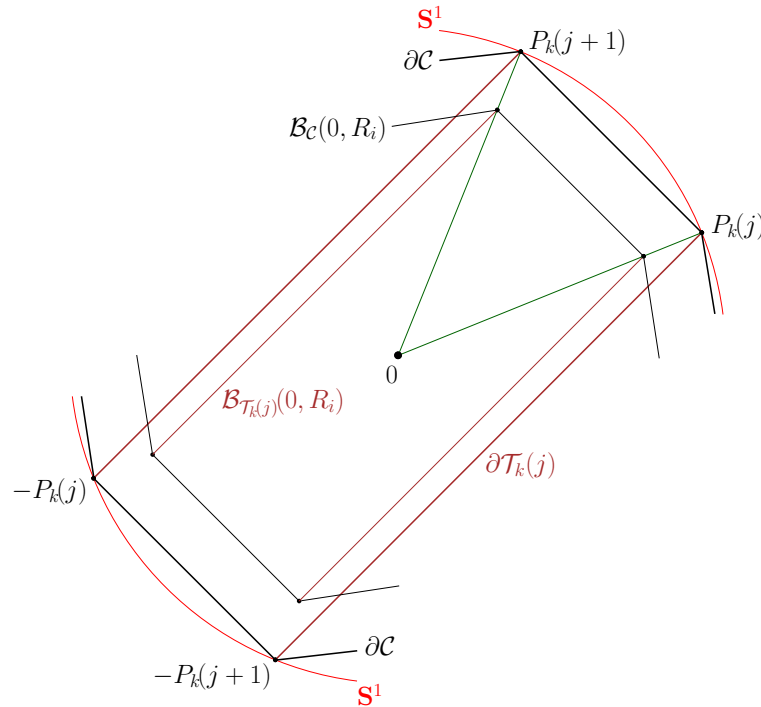


FIGURE 9. Comparing  $\mathcal{B}_{\mathcal{C}}(0, R_i) \cap P_k(j)0P_k(j+1)$  and  $\mathcal{B}_{\mathcal{T}_k(j)}(0, R_i) \cap P_k(j)0P_k(j+1)$

•**First step:** For any  $k \geq 0$  and  $j \in \{0, \dots, n_k - 1\}$ , noticing that  $[P_k(j), P_k(j+1)] \subseteq \partial\mathcal{C} \cap \partial\mathcal{T}_k(j)$  and  $[-P_k(j), -P_k(j+1)] \subseteq \partial\mathcal{C} \cap \partial\mathcal{T}_k(j)$ , we have

$$\mathcal{B}_C(0, R_i) \cap P_k(j)0P_k(j+1) = \mathcal{B}_{\mathcal{T}_k(j)}(0, R_i) \cap P_k(j)0P_k(j+1)$$

by Lemma 3.1, and hence the inclusions

$$\mathcal{B}_{\mathcal{C}}(0, R_i) \cap P_k(j)0P_k(j+1) \subseteq \mathcal{B}_{\mathcal{T}_k(j)}(0, R_i) \subseteq \mathcal{T}_k(j) \subseteq \mathcal{C}$$

hold.

Using Point (iv) of Proposition 5 in [7], we then get

Now, if  $f$  denotes the unique linear transformation of  $\mathbf{R}^2$  such that  $f(P_k(j)) = (1, -1)$  and  $f(P_k(j+1)) = (1, 1)$ , we have

$$f(\mathcal{T}_k(j)) = \mathcal{Q} = (-1, 1) \times (-1, 1) \subseteq \mathbf{R}^2 \quad (\text{standard open square}).$$

The cross ratio being preserved by the linear group  $\text{GL}(\mathbf{R}^2)$ , the map  $f$  induces an isometry between the metric spaces  $(\mathcal{T}_k(j), d_{\mathcal{T}_k(j)})$  and  $(\mathcal{Q}, d_{\mathcal{Q}})$  with  $f(0) = 0$ , and hence we obtain  $\mu_{\mathcal{T}_k(j)}(\mathcal{B}_{\mathcal{T}_k(j)}(0, R_i)) = \mu_{\mathcal{Q}}(\mathcal{B}_{\mathcal{Q}}(0, R_i))$ .

But Proposition 6 in [7] yields

$$\mu_{\mathcal{Q}}(\mathcal{B}_{\mathcal{Q}}(0, R_i)) = 8 \int_0^{\tanh(R_i)} \left( \int_0^x \frac{\pi}{\text{vol}(\mathcal{B}_{\mathcal{C}}(x, y))} dy \right) dx \leq 4 \int_0^{\tanh(R_i)} \left( \int_0^x \frac{\pi}{(1-x^2)(1-y^2)} dy \right) dx = 2\pi R_i^2,$$

which gives

$$(4.5) \quad \mu_{\mathcal{C}}(\mathcal{B}_{\mathcal{C}}(0, R_i) \cap P_k(j)0P_k(j+1)) \leq 2\pi R_i^2$$

from Equation 4.4.

For the same reasons, we also have

$$(4.6) \quad \mu_{\mathcal{C}}(\mathcal{B}_{\mathcal{C}}(0, R_i) \cap -M_{\infty}0M_0) \leq 2\pi R_i^2.$$

•**Second step:** Considering the proof of Theorem 12 in [6], we have

$$\frac{1}{\text{vol}(\mathcal{B}_{\mathcal{C}}(p))} \leq \frac{e^{8R_i}}{\text{vol}(\mathcal{B}_{\mathcal{C}}(0, R_i))}$$

for all  $p \in \mathcal{B}_{\mathcal{C}}(0, R_i)$ .

Therefore, the following is true:

$$(4.7) \quad \begin{aligned} \mu_{\mathcal{C}}(\mathcal{B}_{\mathcal{C}}(0, R_i) \cap S(M_{i+1}0M_{\infty})) &\leq \frac{\pi e^{8R_i}}{\text{vol}(\mathcal{B}_{\mathcal{C}}(0, R_i))} \text{vol}(\mathcal{B}_{\mathcal{C}}(0, R_i) \cap S(M_{i+1}0M_{\infty})) \\ &\leq \frac{\pi e^{8R_i}}{\text{vol}(\mathcal{B}_{\mathcal{C}}(0, 1))} \text{vol}(\mathbf{B}^2 \cap S(M_{i+1}0M_{\infty})) \\ &\quad (\text{using } \mathcal{B}_{\mathcal{C}}(0, 1) \subseteq \mathcal{B}_{\mathcal{C}}(0, R_i) \subseteq \mathbf{B}^2) \\ &= \frac{\pi e^{8R_i}}{\text{vol}(\mathcal{B}_{\mathcal{C}}(0, 1))} \sum_{k=i+1}^{+\infty} \sin(\alpha_k) \\ &\leq \frac{\pi e^{8R_i}}{\text{vol}(\mathcal{B}_{\mathcal{C}}(0, 1))} \sum_{k=i+1}^{+\infty} \alpha_k = \tau \sum_{k=i}^{+\infty} \frac{e^{8R_i}}{n_{k+1}}, \end{aligned}$$

where  $\tau = 2\pi^2/\text{vol}(\mathcal{B}_{\mathcal{C}}(0, 1))$  is a positive constant.

But for all  $k \geq i$  we have

$$e^{8R_i}/n_{k+1} = e^{8R_i} 3^{-n_k^2} \leq 3^{8R_i} 3^{-n_k^2} = 3^{8n_i - n_k^2} = 3^{-n_k^2(1-8n_i/n_k^2)}$$

with  $n_i/n_k^2 \leq 1/n_i$  from the monotone increasing of the sequence  $(n_k)_{k \geq 0}$ .

Hence, since  $1/n_i \rightarrow 0$  as  $i \rightarrow +\infty$ , there exists an integer  $i_0 \geq 0$  such that for all  $k \geq i$  one has  $e^{8R_i}/n_{k+1} \leq 3^{-n_k^2/2}$  whenever  $i \geq i_0$ .

Equation 4.7 then implies

$$\mu_{\mathcal{C}}(\mathcal{B}_{\mathcal{C}}(0, R_i) \cap S(M_{i+1}0M_{\infty})) \leq \tau \sum_{k=i}^{+\infty} 1/3^{n_k^2/2} \leq \tau \sum_{k=i}^{+\infty} (1/3)^k = (3/2)\tau(1/3)^i$$

for all  $i \geq i_0$  (notice that we have  $n^2/2 > 0^\ell > \ell$  for all  $\ell \geq 0$ ; see the second remark following

Now we have  $(1/3)^i \rightarrow 0$  as  $i \rightarrow +\infty$ , and thus there exists an integer  $i_1 \geq i_0$  such that for all  $i \geq i_1$  one has

$$(4.8) \quad \mu_C(\mathcal{B}_C(0, R_i) \cap S(M_{i+1}0M_\infty)) \leq 1.$$

•**Third step:** Combining Equations 4.3, 4.5, 4.6 and 4.8, we eventually get

$$\begin{aligned} \mu_C(\mathcal{B}_C(0, R_i)) &\leq 4\pi R_i^2 + 4\pi R_i^2 \sum_{k=0}^i n_k + 1 \\ &\leq 4\pi R_i^2 + 4\pi R_i^2 (i+1)n_i + 1 \\ &\quad (\text{since the sequence } (n_k)_{k \geq 0} \text{ is non-decreasing}) \\ &= 4\pi R_i^2 + 4\pi (i+1)R_i^3 + 1 \\ &\leq 12\pi R_i^4 \end{aligned}$$

for all  $i \geq i_1$  (since we have  $R_\ell = n_\ell \geq 3^{\ell+1} \geq \ell+1 \geq 1$  for every  $\ell \geq 0$ ), and hence

$$\frac{\ln[\mu_C(\mathcal{B}_C(0, 1))]}{R_i} \leq \frac{\ln[\mu_C(\mathcal{B}_C(0, R_i))]}{R_i} \leq \frac{\ln(12\pi R_i^4)}{R_i},$$

which yields  $\frac{\ln[\mu_C(\mathcal{B}_C(0, R_i))]}{R_i} \rightarrow 0$  as  $i \rightarrow +\infty$ .

This proves the second point of Theorem 4.1.  $\square$

**Remark.** Considering the proof of Point (1) in Theorem 4.1, we can observe that the conclusion  $h(\mathcal{C}) > 0$  we obtained is actually true for *any* sequence of positive integers  $(n_k)_{k \geq 0}$  provided the sequence  $(\theta_k)_{k \in \mathbb{N}}$  converges to some real number  $\theta_\infty$  which satisfies  $0 < \theta_\infty < \pi$ .

## REFERENCES

- [1] BERCK, G., BERNIG, A., AND VERNICOS, C. Volume entropy of Hilbert geometries. *Pac. J. Math.* 245, 2 (2010), 201–225.
- [2] BERNIG, A. Hilbert geometry of polytopes. *Arch. Math.* 92 (2009), 314–324.
- [3] BURAGO, D., BURAGO, Y., AND IVANOV, S. *A course in metric geometry*. AMS, 2001.
- [4] BUSEMANN, H. *The geometry of geodesics*. Academic Press, 1955.
- [5] BUSEMANN, H., AND KELLY, P. *Projective geometry and projective metrics*. Academic Press, 1953.
- [6] COLBOIS, B., AND VERNICOS, C. Bas du spectre et delta-hyperbolicité en géométrie de Hilbert plane. *Bull. Soc. Math. Fr.* 134, 3 (2006), 357–381.
- [7] COLBOIS, B., VERNICOS, C., AND VEROVIC, P. L’aire des triangles idéaux en géométrie de Hilbert. *Enseign. Math.* 50, 3–4 (2004), 203–237.
- [8] COLBOIS, B., VERNICOS, C., AND VEROVIC, P. Hilbert geometry for convex polygonal domains. *J. Geom.* 100 (2011), 37–64.
- [9] COLBOIS, B., AND VEROVIC, P. Hilbert geometry for strictly convex domains. *Geom. Dedicata* 105 (2004), 29–42.
- [10] CRAMPON, M. Entropies of compact strictly convex projective manifolds. *JMD* 3, 4 (2009), 511–547.
- [11] EGLOFF, D. Uniform Finsler Hadamard manifolds. *Ann. Inst. Henri Poincaré, Phys. Théor.* 66 (1997), 323–357.
- [12] GOLDMAN, W. Projective geometry on manifolds. Lecture notes, University of Maryland, 1988.
- [13] SOCIÉ-MÉTHOU, É. *Comportements asymptotiques et rigidités des géométries de Hilbert*. PhD thesis, University of Strasbourg, 2000.
- [14] VERNICOS, C. Introduction aux géométries de Hilbert. *Sémin. Théor. Spectr. Géom.* 23 (2005), 145–168.
- [15] VERNICOS, C. Spectral properties of Hilbert geometries. *Ann. Inst. Math.* 25, 4 (2009), 1143–1168.

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